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# An Upper Bound on the Critical Temperature for a Continuous System with Short-Range Interaction

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A classical gas with short-range interaction in the grand canonical ensemble is studied. If $p(\beta, z)$ denotes the thermodynamic pressure at inverse temperature $\beta$ and activity $z$, then it follows from the Mayer expansion that $p(\beta, z)$ is infinitely differentiable provided $\beta$ and $\beta z$ are sufficiently small. Here it is shown that there exists $\beta_0 > 0$ such that $p(\beta, z)$ is infinitely differentiable if $\beta < \beta_0$ and $z > 0$. One can interpret this result as saying that $(\beta_0)^{-1}$ is an upper bound on the critical temperature for the system.

**KEY WORDS:** Debye screening; cluster expansion, stability of matter.

## 1. INTRODUCTION

We are interested in the thermodynamic behavior of a classical system of particles in three dimensions which interact under a short-range potential. We shall be concerned with the problem of showing that there is a temperature $T_0$ such that if $T > T_0$ then there is no phase transition. We shall interpret this in the following restricted way: Let $p = p(\beta, z)$ be the thermodynamic pressure as a function of the inverse temperature $\beta > 0$ and the activity $z > 0$. Our problem is to show that there is a $\beta_0 > 0$ such that $p(\beta, z)$ is infinitely differentiable if $\beta < \beta_0$ and $z > 0$.

Let us suppose the interaction potential for our system is $v(x, y) = \Phi(x - y)$, where $\Phi$ satisfies the short-range condition

$$|\Phi(x)| \leqslant c/|x|^{3+\varepsilon}, \qquad |x| \geqslant R_0 > 0, \quad \varepsilon > 0 \tag{1.1}$$

and the stability bound

$$\sum_{1 \leqslant i < j \leqslant N} \Phi(x_i - x_j) \geqslant -NB, \qquad B \geqslant 0 \tag{1.2}$$

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The grand partition function in a cube $\Lambda$ at inverse temperature $\beta$ and activity $z$ is

$$Z_G(\beta, z, \Lambda) = \sum_{N=0}^{\infty} \frac{z^N}{N!} \int_{\Lambda^N} \exp\left[ -\beta \sum_{1 \leqslant i < j \leqslant N} v(x_i, x_j) \right] dx_1 \cdots dx_N \qquad (1.3)$$

and the corresponding pressure $p_\Lambda(\beta, z)$ is defined by

$$\beta p_\Lambda(\beta, z) = (\text{Vol } \Lambda)^{-1} \log Z_G(\beta, z, \Lambda) \qquad (1.4)$$

It is well known[11] that if $\Phi$ satisfies the conditions (1.1) and (1.2), then the thermodynamic pressure $p(\beta, z)$ defined by

$$p(\beta, z) = \lim_{\Lambda \to \infty} p_\Lambda(\beta, z) \qquad (1.5)$$

exists and that $p(\beta, z)$ is a continuous function satisfying certain convexity conditions corresponding to thermodynamic stability.

Our goal here is to understand some differentiability properties of $p(\beta, z)$. As a first step in this direction, we consider the standard Mayer expansion for $p(\beta, z)$,[1]

$$\beta p(\beta, z) = \sum_{n=1}^{\infty} a_n(\beta) z^n \qquad (1.6)$$

where $a_1(\beta) = 1$ and

$$|a_n(\beta)| \leqslant [\beta \, \|\Phi\|_1]^{n-1} \, n^{n-2} e^{n\beta B}/n!, \qquad n \geqslant 2 \qquad (1.7)$$

where $\|\Phi\|_1$ is the $L^1$ norm of $\Phi$. It is clear from (1.7) that if

$$\beta \, |z| \cdot \|\Phi\|_1 \, e^{\beta B + 1} < 1 \qquad (1.8)$$

then the series (1.6) converges and is indeed analytic in $z$. In fact, one can easily further show that if $\beta$ and $z$ satisfy (1.8), then $p(\beta, z)$ is infinitely differentiable in $\beta$ and $z$.

The problem with the condition (1.8) is that no matter how small $\beta > 0$ is, there is always a $z > 0$ which violates the inequality. In order to make further progress, we need to specialize to a particular potential $\Phi(x)$ given by

$$v(x, y) = \Phi(x - y) = \frac{1}{8\pi} e^{-|x-y|} = (-\Delta + 1)^{-2} (x, y) \qquad (1.9)$$

where $\Delta$ denotes the three-dimensional Laplace operator. It will be important for us in the following to use the fact that $\Phi$ is the inverse of a local operator. We make a final simplification in our problem by introducing charges into the system. Thus, we assume that the particle at $x_i \in R^3$ has charge $\mathbf{e}_i = \pm 1$ with equal probability 1/2. The grand partition function of the charged system is

$$Z_{C,G}(\beta, z, \Lambda) = \sum_{N=0}^{\infty} \frac{z^N}{N!} \int_{\Lambda^N} \exp\left[ -\beta \sum_{1 \leqslant i < j \leqslant N} \mathbf{e}_i \mathbf{e}_j v(x_i, x_j) \right] dx_1 \cdots dx_N \tag{1.10}$$

where the integration $dx_i$ denotes now integration over $x_i \in \Lambda$ and summation over $\mathbf{e}_i = \pm 1$ with probability 1/2. The pressure $p_{C,\Lambda}(\beta, z)$ corresponding to (1.10) is

$$\beta p_{C,\Lambda}(\beta, z) = (\text{Vol } \Lambda)^{-1} \log Z_{C,G}(\beta, z, \Lambda) \tag{1.11}$$

and the thermodynamic pressure $p_C(\beta, z)$ for the charged system is again

$$p_C(\beta, z) = \lim_{\Lambda \to \infty} p_{C,\Lambda}(\beta, z) \tag{1.12}$$

We then have the following result.

**Theorem 1.1.** There is a $\beta_0 > 0$ such that $p_C(\beta, z)$ is infinitely differentiable for all $(\beta, z)$ such that $0 < \beta < \beta_0$, $z > 0$.

A result of a similar nature has been proven for a lattice Coulomb gas by Fröhlich and Spencer[8] with techniques different from those used here.

For the potential $\Phi$ given by (1.9), $\|\Phi\|_1 = 1$, $B = 1/16\pi$. Hence, if we put

$$\hat{z} = z \exp(\beta/16\pi) \tag{1.13}$$

then from (1.8) we see that $p_C(\beta, z)$ is infinitely differentiable if

$$|\beta \hat{z}| < \mathbf{e}^{-1} \tag{1.14}$$

Now let us suppose $z > 0$ and define a correlation length $l_c = (\beta \hat{z})^{-1/4}$. Hence the condition (1.14) is $l_c > \mathbf{e}^{1/4}$. We shall prove the following.

**Theorem 1.2.** Suppose $l_c \leqslant 2$. Then there is an $\varepsilon > 0$ such that $p_C(\beta, z)$ is infinitely differentiable provided $\beta l_c < \varepsilon$.

It is easy to derive Theorem 1.1 from Theorem 1.2 and (1.14). From (1.14) it is sufficient to prove the theorem when $l_c \leqslant 2$. In that case we have $\hat{z} \geqslant 1/16\beta$, and Theorem 1.2 tells us that $p_C(\beta, z)$ is infinitely differentiable if $\beta l_c < \varepsilon$. This latter condition is the same as $\hat{z} > \beta^3/\varepsilon^4$. Hence, $p_C(\beta, z)$ is infinitely differentiable in $(\beta, z)$ for all $z > 0$ provided $\beta^3/\varepsilon^4 < 1/16\beta$. We can therefore take $\beta_0 = \varepsilon/2$ in Theorem 1.1.

The condition $\beta l_c$ small of Theorem 1.2 says that in a perfect gas with activity $\hat{z}$ there are many particles in a cube with side of length $l_c$. In fact, the expectation of the number of particles inside such a cube is $\hat{z} l_c^3 = 1/\beta l_c$. Thus, we expect that, for our charged system, the charge on a cube with side of length $l_c$ is approximately Gaussian with mean zero provided $\beta l_c$ is small. By integrating these Gaussian variables out of the system, we obtain a partition function which has a convergent Mayer series and it is this which yields Theorem 1.2. The scheme does not work for the repulsive system with potential (1.9). Although the number of particles in a cube with side of length $l_c$ is still approximately Gaussian, the mean becomes large as $\beta l_c$ becomes small. This problem seems to be related to the multiscale problems studied by Gawędzki and Kupiainen.[9]

It is easy to obtain from the present techniques a result similar to Theorem 1.1 for the differentiability of $P_c$ as a function of $\beta$ and *density* $\rho$,

$$\rho = z\beta \frac{\partial p_c}{\partial z} \tag{1.15}$$

The reason is that when $z$ satisfies (1.14), the Mayer expansion (1.6) holds, while in the regime of Theorem 1.2 the expansion (2.8), (3.21) holds. In particular there are constants $c_1$, $c_2 > 0$ such that

$$c_1 \leqslant \frac{\partial p}{\partial z} \leqslant c_2, \quad z > 0 \tag{1.16}$$

It follows then that Eq. (1.15) can be inverted to obtain $z$ as a function of $\beta$ and $\rho$. One should perhaps note here that our expansions do not yield $p_c(\beta, z)$ as an analytic function of $z$ when $z$ is large.

To implement the present scheme, we shall adapt the method developed by Brydges and Federbush[2] to prove Debye screening in classical Coulomb systems. This method has been developed in great generality in refs. 2, 7, 10, and 12. Here we shall only be concerned with the basic technique in its simplest form. We do some estimates rather differently than in ref. 2, using a lemma of Federbush[6] which has been used in problems concerned with the stability of matter.[4–6] The remainder of this paper shall be devoted to the proof of Theorem 1.2.

## 2. FIELD-THEORETIC FORMULATION

Here we follow the ideas of ref. 2, and write the partition function (1.10) in the sine-Gordon representation. Let $\Delta_D$ be the Laplace operator with Dirichlet boundary conditions imposed on $\partial\Lambda$. Then the potential (1.9) with Dirichlet boundary conditions imposed is $v_D(x, y)$, where

$$v_D(x, y) = (-\Delta_D + 1)^{-2}(x, y) \tag{2.1}$$

For $x \in \Lambda$ let us define $z(x)$ by

$$z(x) = z \exp\left\{\frac{\beta}{2}[v(x, x) - v_D(x, x)]\right\} \tag{2.2}$$

and the Dirichlet partition function $Z_{C,G,D}$ by

$$Z_{C,G,D}(\beta, z, \Lambda) = \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\Lambda^N} \exp\left[-\beta \sum_{1 \leqslant i < j \leqslant N} \mathbf{e}_i \mathbf{e}_j v_D(x_i, x_j)\right] \times z(x_1) \cdots z(x_N)\, dx_1 \cdots dx_N \tag{2.3}$$

The corresponding pressure is $p_{C,D,\Lambda}$, where

$$\beta p_{C,D,\Lambda}(\beta, z) = (\text{Vol }\Lambda)^{-1} \log Z_{C,G,D}(\beta, z, \Lambda) \tag{2.4}$$

Then one can easily prove the following using the techniques in ref. 11.

**Lemma 2.1.** Let $p_C(\beta, z)$ be defined by (1.12). Then

$$p_C(\beta, z) = \lim_{\Lambda \to \infty} p_{C,D,\Lambda}(\beta, z) \tag{2.5}$$

The advantage of the partition function (2.3) over (1.10) is that it has a simpler representation in the sine-Gordon field theory. We have

$$Z_{C,G,D}(\beta, z, \Lambda) = E\left\{\exp\left[\int_\Lambda \hat{z} \cos \beta^{1/2}\phi(x)\, dx\right]\right\} \tag{2.6}$$

where the expectation $E$ is over fields $\phi(x)$ on $\Lambda$ with covariance $v_D(x, y)$. If we put

$$\beta P_\Lambda(\beta, z) = (\text{Vol }\Lambda)^{-1} \log E\left[\exp\left\{\int_\Lambda \hat{z}[\cos \beta^{1/2}\phi(x) - 1]\, dx\right\}\right] \tag{2.7}$$

then we have the relation

$$\beta p_{C,D,\Lambda}(\beta, z) = \hat{z} + \beta P_\Lambda(\beta, z) \tag{2.8}$$

We shall develop a perturbation series expansion for $P_\Lambda(\beta, z)$ similar to ref. 1 by expanding the expression (2.7) with respect to the covariance corresponding to the differential operator $\mathfrak{L}_D$, where

$$\mathfrak{L}_D = (-\Delta_D + 1)^2 + 1/l_c^4 \tag{2.9}$$

which is obtained from the sine-Gordon representation (2.7) by making the approximation

$$\hat{z}[\cos \beta^{1/2}\phi(x) - 1] \simeq -(\hat{z}\beta)\, \phi(x)^2/2 \tag{2.10}$$

To implement this, we cover $R^3$ with a lattice of cubes $Q$ with side of length $Ll_c$, where $0 < L < 1$, and $L$ shall be chosen appropriately small later to make the perturbation series converge. We shall assume that the large cube $\Lambda$ is always a union of the small cubes $Q$. Next let $h(x)$ be functions on $R^3$, constant on cubes $Q$, such that $h(x) = 0$ if $x \notin \Lambda$ and $\beta^{1/2}h(x)/2\pi$ is integer valued. We define a function $G(\phi)$ of the field $\phi$ by the equation

$$\exp\left\{\int_\Lambda \hat{z}[\cos \beta^{1/2}\phi(x) - 1]\, dx\right\} = e^{G(\phi)} \sum_h \exp\left\{\frac{-1}{2l_c^4}\int_\Lambda [\phi(x) - h(x)]^2\, dx\right\} \tag{2.11}$$

Let $g$ be a linear mapping on the functions $h$ given by

$$g(h) = \frac{1}{l_c^4}\, \mathfrak{L}_D^{-1} h \tag{2.12}$$

For two functions $h$ and $h'$ we define an inner product $F(h, h')$ by the identity

$$2F(h, h') = \langle g(h), (-\Delta_D + 1)^2\, g(h')\rangle + \frac{1}{l_c^4}\langle g(h) - h, g(h') - h'\rangle \tag{2.13}$$

where $\langle\ ,\ \rangle$ denotes the $L^2$ inner product on $\Lambda$. If we put $F(h) = F(h, h)$, then, on making the translation $\phi = \psi + g(h)$, we see that

$$\beta P_\Lambda(\beta, z) = (\mathrm{Vol}\, \Lambda)^{-1} \log \kappa(\mathfrak{L}_D) + (\mathrm{Vol}\, \Lambda)^{-1} \log\left(\sum_h \exp[-F(h)]\, E_{\mathfrak{L}_D}\{\exp[G(\psi + g(h))]\}\right) \tag{2.14}$$

where $\kappa(\mathfrak{L}_D)$ is given by

$$\kappa(\mathfrak{L}_D) = \det\left[I + \frac{1}{l_c^4}(-\Delta_D + 1)^{-2}\right]^{-1/2} \tag{2.15}$$

and $E_{\mathfrak{L}_D}$ denotes expectation of fields $\psi(x)$ on $\Lambda$ with covariance $\mathfrak{L}_D^{-1}$. Writing $C_D(x, y)$ as the kernel of the operator $\mathfrak{L}_D^{-1}$, we have the following inequality.

**Lemma 2.1.** Suppose $l_c \leqslant 2$. Then there are universal constants $A$, $\lambda > 0$ such that

$$\begin{aligned} |C_D(x, y)| &\leqslant A l_c \exp(-|x-y|/\lambda l_c) \\ |\nabla_x C_D(x, y)| &\leqslant A \exp(-|x-y|/\lambda l_c) \\ |\nabla_x \nabla_y C_D(x, y)| &\leqslant A l_c^{-1} \exp(-|x-y|/\lambda l_c) \end{aligned} \tag{2.16}$$

*Proof.* We consider the operator $\mathfrak{L}$ on $R^3$ given by $\mathfrak{L} = (-\Delta + 1)^2 + 1/l_c^4$. Then we evidently have the identity

$$\mathfrak{L}^{-1} = \frac{l_c^2}{2i}\left[\left(-\Delta + 1 - \frac{i}{l_c^2}\right)^{-1} - \left(-\Delta + 1 + \frac{i}{l_c^2}\right)^{-1}\right] \tag{2.17}$$

We write

$$1 + \frac{i}{l_c^2} = \left(1 + \frac{1}{l_c^4}\right)^{1/2} \mathrm{e}^{i\theta} \tag{2.18}$$

where $0 < \theta < \pi/2$, $\tan\theta = 1/l_c^2$. Then the kernel of $\mathfrak{L}^{-1}$ is given by the formula

$$\begin{aligned} \mathfrak{L}^{-1}(x, y) &= \frac{l_c^2}{8\pi i\,|x-y|}\left\{\exp\left[-\left(1+\frac{1}{l_c^4}\right)^{1/4} \mathrm{e}^{-i\theta/2}\,|x-y|\right]\right. \\ &\quad \left. - \exp\left[-\left(1+\frac{1}{l_c^4}\right)^{1/4} \mathrm{e}^{i\theta/2}|x-y|\right]\right\} \\ &= \frac{l_c^2}{4\pi\,|x-y|} \exp\left[-\left(1+\frac{1}{l_c^4}\right)^{1/4} \cos\frac{\theta}{2}\,|x-y|\right] \\ &\quad \times \sin\left[\left(1+\frac{1}{l_c^4}\right)^{1/4} \sin\frac{\theta}{2}\,|x-y|\right] \end{aligned} \tag{2.19}$$

Using the fact that $l_c \leqslant 2$, $0 < \theta/2 < \pi/4$, it is easy to see that

$$|\mathfrak{L}^{-1}(x, y)| \leqslant \frac{17^{1/4}}{4\pi\sqrt{2}}\, l_c \exp(-|x-y|/\sqrt{2}\, l_c)$$

The inequality (2.16) follows easily from (2.20), since one can construct $C_D(x, y)$ by the method of images from $\mathfrak{L}^{-1}(x, y)$.

## 3. THE CLUSTER EXPANSION

Our goal here is to develop a cluster expansion for the partition function given in (2.14). First, for a lattice cube $Q \subset \Lambda$ let $G_Q(\phi)$ be defined by

$$\exp[G_Q(\phi)] = \exp\left\{\int_Q \hat{z}[\cos \beta^{1/2}\phi(x) - 1]\, dx\right\} \times \left\{\sum_{n=-\infty}^{\infty} \exp\left[\frac{-1}{2l_c^4}\int_Q \left(\phi(x) - \frac{2\pi n}{\sqrt{\beta}}\right)^2 dx\right]\right\}^{-1} \tag{3.1}$$

If $X \subset \Lambda$ is a union of lattice cubes $Q$, we define $G_X(\phi)$ by

$$G_X(\phi) = \sum_{Q \subset X} G_Q(\phi) \tag{3.2}$$

and it is evident that $G(\phi)$ defined by (2.11) is identical to $G_\Lambda(\phi)$.

Let $h(x)$ be a discrete-valued function occurring in the definition of the pressure (2.14), and suppose $\Sigma(h)$ is the discontinuity set for $h$. Thus, $\Sigma(h) \subset \Lambda$ and $\Sigma(h)$ consists of a set of faces of lattice cubes $Q$. Suppose $X \subset \Lambda$ is a connected set consisting of a union of cubes $Q$. We denote by $h_X$ any function on $R^3$ constant on cubes $Q$ with $\beta^{1/2}h_X/2\pi$ integer valued such that $\Sigma(h_X) \subset \mathrm{Int}[X \cup R^3 \backslash \Lambda]$, $h_X(x) = 0$ if $x \in R^3$ is large, and every cube $Q \subset X$ has nonempty intersection with $\Sigma(h_X)$. It is easy to see then that every $h$ function has a unique decomposition

$$h = \sum_{i=1}^{n} h_{X_i} \tag{3.3}$$

where the connected sets $X_i$, $1 \leqslant i \leqslant n$, are all disjoint and contained in $\Lambda$.

Suppose $X$ is a subset of $\Lambda$ which is a union of lattice cubes $Q$. For $i = 1, 2, \ldots$ we shall define the activity $K_i(X)$ of $i$-particle clusters on $X$. First we take the case $i = 1$. When $X = Q$ we define $K_1(Q)$ by

$$K_1(Q) = E_{\mathfrak{Q}_D}\{\exp[G_Q(\psi)] - 1\} \tag{3.4}$$

If $X$ is disconnected, we put $K_1(X) = 0$ and if $X \neq Q$ is connected, we define $K_1(X)$ by

$$K_1(X) = \sum_{h_X} \exp[-F(h_X)]\, E_{\mathfrak{Q}_D}\{\exp[G_X(\psi + g(h_X))]\} \tag{3.5}$$

For $i = 2$, we put $K_2(Q) = 0$. Letting $|X|$ be the number of lattice cubes contained in the set $X$, we define $K_2(X)$ for $|X| \geqslant 2$ as a sum

$$K_2(X) = \sum_{Y \cup Z = X} K_2(Y, Z) \tag{3.6}$$

The sum is taken over disjoint sets $Y$, $Z$ whose union is $X$ and which themselves are unions of lattice cubes. The quantity $K_2(Y, Z)$ is defined differently according to the type of sets $Y$, $Z$. Let us suppose that both $Y$ and $Z$ are lattice cubes. Then $K_2(Y, Z)$ is defined by interpolating the covariance $C_D(x, y)$. For $0 \leqslant s \leqslant 1$ let $\mathfrak{L}_{D,s}$ be the operator with covariance

$$C_{D,s}(x, y) = \{s + (1-s)[\chi_Y(x)\,\chi_Y(y) + \chi_Z(x)\,\chi_Z(y)]\}\; C_D(x, y) \tag{3.7}$$

Then we have

$$K_2(Y, Z) = \int_0^1 ds E_{\mathfrak{L}_{D,s}} \left\{ \int_Y \int_Z C_D(x, y) \frac{\delta}{\delta\psi(x)} \frac{\delta}{\delta\psi(y)}\, dx\, dy \exp[G_X(\psi)] \right\} \tag{3.8}$$

Next suppose $Z$ is a lattice cube but $Y$ is connected and $|Y| > 1$. If there exists a function $h_Y$ we need to interpolate $g(h_Y)$ as well as the covariance. For $0 \leqslant s \leqslant 1$, let $g_s(x)$ be defined by

$$\begin{aligned} g_s(x) &= g(h_Y)(x), \qquad && x \in Y \\ &= sg(h_Y)(x), \qquad && x \notin Y \end{aligned} \tag{3.9}$$

Then $K_2(Y, Z)$ in this case is given by

$$\begin{aligned} K_2(Y, Z) = \int_0^1 ds \sum_{h_Y} \exp[-F(h_Y)]\, E_{\mathfrak{L}_{D,s}} \\ \times \left\{ \left[ \int_Y \int_Z C_D(x, y) \frac{\delta}{\delta\psi(x)} \frac{\delta}{\delta\psi(y)}\, dx\, dy \right.\right. \\ \left.\left. + \int_Z g(h_Y)(x) \frac{\delta}{\delta\psi(x)}\, dx \right] \exp[G_X(\psi + g_s)] \right\} \end{aligned} \tag{3.10}$$

If both $Y$ and $Z$ are connected with $|Y| > 1$ and $|Z| > 1$, and there exist an $h_Y$ and $h_Z$, then we also need to interpolate $F(h_Y + h_Z)$. For $0 \leqslant s \leqslant 1$, let $F_s$ be defined by

$$F_s = F(h_Y) + F(h_Z) + 2sF(h_Y, h_Z) \tag{3.11}$$

We interpolate $g(h_Y + h_Z)$ as follows:

$$\begin{aligned} g_s(x) &= sg(h_Y + h_Z)(x) + (1-s)\; g(h_Y)(x), \qquad && x \in Y \\ &= sg(h_Y + h_Z)(x) + (1-s)\; g(h_Z)(x), \qquad && x \in Z \\ &= sg(h_Y + h_Z)(x), \qquad && x \notin Y \cup Z \end{aligned} \tag{3.12}$$

Then $K_2(Y, Z)$ is given by the formula

$$K_2(Y, Z) = \int_0^1 ds \sum_{h_Y, h_Z} \exp(-F_s)\, E_{\mathfrak{L}_{D,s}}$$
$$\times \left\{ \left[ -2F(h_Y, h_Z) + \int_Y \int_Z C_D(x, y) \frac{\delta}{\delta\psi(x)} \frac{\delta}{\delta\psi(y)}\, dx\, dy \right.\right.$$
$$\left. + \int_Y g(h_Z)(x) \frac{\delta}{\delta\psi(x)}\, dx + \int_Z g(h_Y)(x) \frac{\delta}{\delta\psi(x)}\, dx \right]$$
$$\left. \times \exp[G_X(\psi + g_s)] \right\} \tag{3.13}$$

For all other subdivisions $Y$ and $Z$ of $X$ not included in the above three categories we put $K_2(Y, Z) = 0$.

The $n$-particle activity on $X$, $K_n(X)$, is defined for $n \geqslant 2$ in a similar way to the $n = 2$ case. First $K_n(X) = 0$ if $|X| < n$. For $|X| \geqslant n$, $K_n(X)$ is given as a sum,

$$K_n(X) = \sum_{\bigcup_{i=1}^n X_i = X} K_n(X_1, X_2, ..., X_n) \tag{3.14}$$

The sum in (3.14) is over disjoint sets $X_i$, $1 \leqslant i \leqslant n$, whose union is $X$ and which themselves are unions of lattice cubes. In addition, the sum is taken over *distinct* sets $\{X_1, X_2, ..., X_n\}$ so that permutations of the $X_i$ are not counted in (3.14). To define $K_n(X_1, ..., X_n)$, we fix a tree graph $T$ on the integers 1, 2,..., $n$. Let $\mathbf{s}$ be a parameter which varies in the set $\Gamma_n = [0, 1]^{n-1}$ and $J$ be a partition of the integers 1, 2,..., $n$. For a given $\mathbf{s}$, interpolation is constructed with parameters $\lambda_{T,J}(\mathbf{s}) \geqslant 0$, where

$$\sum_J \lambda_{T,J}(\mathbf{s}) = 1 \tag{3.15}$$

The interpolated covariance $C_{D,\mathbf{s}}(x, y)$ is given by

$$C_{D,\mathbf{s}}(x, y) = C_D(x, y) \sum_J \lambda_{T,J}(\mathbf{s}) \sum_{S \in J} \chi_S(x)\, \chi_S(y) \tag{3.16}$$

where $\chi_S$ denotes the characteristic function of the set $\bigcup_{i \in S} X_i$. The interpolated function $F$, denoted $F_\mathbf{s}$, is given by

$$F_\mathbf{s} = \sum_J \lambda_{T,J}(\mathbf{s}) \sum_{S \in J} F\left( \sum_{i \in S} h_{X_i} \right) \tag{3.17}$$

The interpolated $g$, denoted $g_\mathbf{s}$, is defined by

$$g_\mathbf{s}(x) = \sum_J \lambda_{T,J}(\mathbf{s})\, g\left( \sum_{i \in S(J,k)} h_{X_i} \right)(x), \qquad x \in X_k \tag{3.18}$$

where $S(J, k)$ denotes the set in $J$ which includes the integer $k$. There is then a probability measure $d\mu_T$ on $\Gamma_n$ such that

$$\begin{aligned}K_n(X_1, ..., X_n) = \sum_T \int_{\Gamma_n} d\mu_T(\mathbf{s}) \sum_{h_{X_i}} \exp(-F_{\mathbf{s}}) \\ \times E_{\mathfrak{L}_{D,\mathbf{s}}}\Bigg[ \prod_{(i,j)\in T} \Bigg\{ -2F(h_{X_i}, h_{X_j}) \\ + \int_{X_i}\int_{X_j} C_D(x, y) \frac{\delta}{\delta\psi(x)} \frac{\delta}{\delta\psi(y)} dx\, dy \\ + \int_{X_i} g(h_{X_j})(x) \frac{\delta}{\delta\psi(x)} dx + \int_{X_j} g(h_{X_i})(x) \frac{\delta}{\delta\psi(x)} dx \Bigg\} \\ \times \exp[G_X(\psi + g_{\mathbf{s}})] \Bigg]\end{aligned} \tag{3.19}$$

For the sum in (3.19) over $h_{X_i}$ we include $h_{X_i} \equiv 0$ when $X_i$ is a lattice cube. The formula (3.19) clearly generalizes the previous formulas (3.8), (3.10), and (3.13) for the case $n = 2$.

The total activity of a set $X$ is given by $K(X)$, where

$$K(X) = \sum_{n=1}^{\infty} K_n(X) \tag{3.20}$$

The pressure $P_\Lambda$ in (2.14) is then given by the formula

$$\begin{aligned}\beta P_\Lambda(\beta, z) = (\mathrm{Vol}\,\Lambda)^{-1} \log \kappa(\mathfrak{L}_D) \\ + (\mathrm{Vol}\,\Lambda)^{-1} \log\left[1 + \sum_{m=1}^{\infty} K(Y_1)\cdots K(Y_m)\right]\end{aligned} \tag{3.21}$$

where the summation is over disjoint subsets $Y_i$ of $\Lambda$, $1 \leqslant i \leqslant m$, which are unions of lattice cubes. In addition, the sum is over distinct sets $\{Y_1, ..., Y_m\}$. Formula (3.21) is derived in detail in Sections 2.0–2.6 of ref. 3. Equation (3.21) gives us our cluster expansion. The partition function on the right in (3.21) is a hard-core-gas partition function on sets $Y_i$ and can be expanded as in ref. 1. The perturbation series converges if, for a lattice cube $Q$,

$$\sum_{\substack{Q \in Y \\ |Y| = N}} |K(Y)| \leqslant \mathrm{e}^{-cN} \tag{3.22}$$

for sufficiently large constant $c > 0$.

## 4. ESTIMATION OF ONE-PARTICLE ACTIVITIES

Here we estimate the one-particle activities $K_1(X)$ for various sets $X$. First we need to discuss the properties of a certain function of a real variable $r_\eta(A)$ which depends on a small parameter $\eta > 0$. We define $r_\eta(A)$ to be

$$r_\eta(A) = \exp\left(\frac{\cos \eta A - 1}{\eta^2}\right) \Big/ \sum_{n=-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(A - \frac{2\pi n}{\eta}\right)^2\right] \tag{4.1}$$

Evidently $r_\eta(A)$ is periodic with period $2\pi/\eta$, and

$$r_\eta(0) = 1 + O\left(\exp \frac{-\pi^2}{\eta^2}\right) \tag{4.2}$$

Furthermore, it is easy to see that $r_\eta(A)$ is bounded below for all $A$,

$$r_\eta(A) \geqslant 1 + O\left(\exp \frac{-\pi^2}{2\eta^2}\right) \tag{4.3}$$

On the other hand, $r_\eta(A)$ can become arbitrarily large for sufficiently small $\eta$ and in fact

$$r_\eta(\pi/\eta) = O\{\exp[(\pi^2 - 4)/2\eta^2]\} \tag{4.4}$$

We have from ref. 2 the following lemma, which estimates the derivatives of $r_\eta(A)$

**Lemma 4.1**. *There is a number $\gamma$, $0 < \gamma < 1$, and positive constants $c_1$, $c_2$, $c_3$ which are universal such that*

$$\left|\left(\frac{d}{dA}\right)^N r_\eta(A)\right| \leqslant c_1 (c_2 \eta^{1/3})^N \exp[c_2 N \log(N+1)]$$
$$\times \exp[(\gamma A^2/2)], \qquad N \geqslant 0 \tag{4.5}$$

We use the functions $r_\eta(A)$ to estimate $K_1(Q)$ in the following.

**Lemma 4.2.** There is a constant $C$ such that if $L = (\beta l_c)^{1/10}$ and $\beta l_c < 1$, then $|K_1(Q)| < C(\beta l_c)^{1/10}$ for any lattice cube $Q$.

*Proof.* For a function $\psi(x)$ on $\Lambda$, let $P\psi$ be the projection of $\psi$ onto functions constant on lattice cubes. Thus,

$$P\psi(x) = (\text{Vol } Q)^{-1} \int_Q \psi(y)\, dy, \qquad x \in Q \tag{4.6}$$

We write then

$$\exp[G_Q(\psi)] = r_\eta(A)\exp[R_Q(\psi)] \tag{4.7}$$

$$\eta = (\beta l_c)^{1/2}/L^{3/2}, \qquad A = L^{3/2}P\psi(x)/l_c^{1/2}, \qquad x \in Q \tag{4.8}$$

From (3.1) and (4.1) we then have the formula

$$\begin{aligned} \exp[R_Q(\psi)] = \exp\Bigg\{&\frac{1}{2l_c^4}\int_Q [\psi(x) - P\psi(x)]^2\,dx \\ &+ \int_Q \hat{z}[\cos\beta^{1/2}\psi(x) - \cos\beta^{1/2}P\psi(x)]\,dx\Bigg\} \\ = \exp\Bigg\{&\frac{1}{2l_c^4}\int_Q [\psi(x) - P\psi(x)]^2 \\ &\times\left(1 - 2\int_0^1\int_0^1 u\cos\{\beta^{1/2}P\psi(x)\right. \\ &\left. + uv\beta^{1/2}[\psi(x) - P\psi(x)]\}\,du\,dv\right)dx\Bigg\} \end{aligned} \tag{4.9}$$

From (4.9) one easily sees that

$$1 \leqslant \exp[R_Q(\psi)] \leqslant \exp\left\{\frac{1}{l_c^4}\int_Q [\psi(x) - P\psi(x)]^2\,dx\right\} \tag{4.10}$$

Next we write

$$\exp[G_Q(\psi)] = I_1(\psi) + I_2(\psi) + I_3(\psi) \tag{4.11}$$

where

$$\begin{aligned} I_1(\psi) &= [r_\eta(A) - r_\eta(0)]\exp[R_Q(\psi)] \\ I_2(\psi) &= [r_\eta(0) - 1]\exp[R_Q(\psi)] \\ I_3(\psi) &= \exp[R_Q(\psi)] \end{aligned} \tag{4.12}$$

Let $\chi_Q$ be the characteristic function for $A$ and put $N = (2/l_c^4)$ $(I - P)\chi_Q(I - P)$. Then from (4.10) we have the inequality

$$1 \leqslant E_{\mathfrak{L}_D}[I_3(\psi)] \leqslant E_{\mathfrak{L}_D}[\exp(\tfrac{1}{2}\langle\psi, N\psi\rangle)] \tag{4.13}$$

The expectation on the right in (4.13) is a Gaussian integral and can be evaluated explicitly. If we let $M = \mathfrak{L}_D^{-1}$, then

$$E_{\mathfrak{L}_D}[\exp(\tfrac{1}{2}\langle\psi, N\psi\rangle)] = \det(I - M^{1/2}NM^{1/2})^{-1/2} \tag{4.14}$$

Now we have

$$\langle\psi, \mathfrak{L}_D\psi\rangle \geqslant \langle\psi, (-\Delta_D)^2\psi\rangle \geqslant \langle\psi, -\Delta_D\psi\rangle^2/\|\psi\|^2 \tag{4.15}$$

and

$$\langle\psi, -\Delta_D\psi\rangle \geqslant \frac{1}{2}\left(\frac{\pi l_c}{L}\right)^2 \langle\psi, N\psi\rangle \tag{4.16}$$

It follows then, since $\frac{1}{2}\langle\psi, N\psi\rangle \geqslant \|\psi\|^2/l_c^4$, that

$$\langle\psi, \mathfrak{L}_D\psi\rangle \geqslant \tfrac{1}{2}\left(\frac{\pi}{L}\right)^4 \langle\psi, N\psi\rangle \tag{4.17}$$

Hence, if $L < \pi/2$, then $\mathfrak{L}_D \geqslant 8N$, whence one has the inequality

$$M^{1/2}NM^{1/2} \leqslant I/8 \tag{4.18}$$

From (4.18) it follows then that

$$0 \leqslant \log\det(I - M^{1/2}NM^{1/2})^{-1/2} \leqslant \tfrac{4}{7}\operatorname{Tr} M^{1/2}NM^{1/2} \tag{4.19}$$

and one easily sees that

$$\begin{aligned}\operatorname{Tr} M^{1/2}NM^{1/2} &\leqslant \frac{2}{l_c^4}\operatorname{Tr} M^{1/2}\chi_Q M^{1/2}\\ &= \frac{2}{l_c^4}\int_Q C_D(x,x)\,dx \leqslant 2AL^3\end{aligned} \tag{4.20}$$

by Lemma 2.1. We conclude then that

$$1 \leqslant E_{\mathfrak{L}_D}[I_3(\psi)] \leqslant \exp(\tfrac{8}{7}AL^3) \tag{4.21}$$

From (4.2) and (4.21) we have that

$$|E_{\mathfrak{L}_D}[I_2(\psi)]| \leqslant O\left(\exp\frac{-\pi^2}{\eta^2}\right)\exp\left(\frac{8}{7}AL^3\right) \tag{4.22}$$

We use Lemma 4.1 to estimate $I_1(\psi)$. Thus,

$$\begin{aligned}|I_1(\psi)| &\leqslant C\eta^{1/3}\,|A|\exp(\gamma A^2/2 + \tfrac{1}{2}\langle\psi, N\psi\rangle)\\ &\leqslant C'\eta^{1/3}\exp(\gamma' A^2/2 + \tfrac{1}{2}\langle\psi, N\psi\rangle)\end{aligned} \tag{4.23}$$

for any $\gamma' > \gamma$ provided $C'$ is chosen appropriately depending on $\gamma' - \gamma$. Since $\gamma < 1$, we can take $\gamma' < 1$. If we let $W$ be the operator

$$W = \frac{\gamma'}{l_c^4} P\chi_Q P + N \tag{4.24}$$

then it is clear from (4.8) and (4.15) that

$$|I_1(\psi)| \leqslant C'\eta^{1/3} \exp(\tfrac{1}{2}\langle \psi, W\psi \rangle) \tag{4.25}$$

Arguing as before, we have that

$$M^{1/2}WM^{1/2} \leqslant \gamma' I \tag{4.26}$$

assuming $L < \pi/2$ and $\gamma' \geqslant 1/8$. Then

$$0 \leqslant \log \det(I - M^{1/2}WM^{1/2})^{-1/2} \leqslant \frac{1}{2(1-\gamma')} \operatorname{Tr} M^{1/2}WM^{1/2} \tag{4.27}$$

and since $W \leqslant (2/l_c^4)\chi_Q$ we have, as in (4.20),

$$\operatorname{Tr} M^{1/2}WM^{1/2} \leqslant 2AL^3 \tag{4.28}$$

We conclude then that

$$E_{\mathfrak{L}_D}[|I_1(\psi)|] \leqslant C'\eta^{1/3} \exp \frac{AL^3}{1-\gamma'} \tag{4.29}$$

The result follows now from (4.21), (4.22), and (4.29). ∎

We turn next to extimating the one-particle activities $K_1(X)$ for connected sets $X$ which have a function $h_X$ associated with them. To do this, we use a lemma of Federbush[6] which has been useful in problems related to the stability of matter.[4–6]

**Lemma 4.3.** Let $H_D$ be the operator $H_D = (-\Delta_D + 1)^2 + V$ with potential $V(x) \geqslant 0$, $V \in L^\infty$, and suppose $V(x)$ is constant on some open set $U \subset R^3$. Let $\chi_i(x)$, $i = 1, 2, \ldots, m$, be functions in the domain of $(-\Delta_D)^2$ which have support contained in $U$ and such that the supports of the $\chi_i$ are all disjoint, $1 \leqslant i \leqslant m$. Then for any function $h \in L^\infty(\Lambda)$ there is the inequality

$$\begin{aligned} &\langle h, VH_D^{-1}(-\Delta_D + 1)^2 H_D^{-1} Vh \rangle \\ &\qquad \geqslant \sum_{i=1}^{m} \langle Vh, -\Delta_D \chi_i \rangle^2 / \langle \chi_i, H_D^2 \chi_i \rangle \end{aligned} \tag{4.30}$$

*Proof.* Let $\phi_i = H_D \chi_i$, $1 \leqslant i \leqslant m$. Since $\chi_i$ is in the domain of $(-\Delta_D)^2$ then $\phi_i \in L^2(\Lambda)$. Since the supports of the $\chi_i$ do not intersect, it follows that the $\phi_i$ form an orthogonal set, $1 \leqslant i \leqslant m$. Hence we have the inequality

$$\begin{aligned}
&\langle h, VH_D^{-1}(-\Delta_D+1)^2 H_D^{-1}Vh\rangle \\
&\quad \geqslant \langle h, VH_D^{-1}(-\Delta_D)^2 H_D^{-1}Vh\rangle = \|-\Delta_D H_D^{-1}Vh\|^2 \\
&\quad \geqslant \sum_{i=1}^{m} \langle -\Delta_D H_D^{-1}Vh, \phi_i\rangle^2 |\langle \phi_i, \phi_i\rangle \\
&\quad = \sum_{i=1}^{m} \langle Vh, H_D^{-1}(-\Delta_D)\phi_i\rangle^2/\langle \phi_i, \phi_i\rangle
\end{aligned} \tag{4.31}$$

The result now follows by observing that since $V$ is constant on $U$, we have

$$\begin{aligned}
H_D^{-1}(-\Delta_D)\phi_i &= H_D^{-1}(-\Delta_D)\, H_D\chi_i \\
&= H_D^{-1}H_D(-\Delta_D)\chi_i = -\Delta_D\chi_i \quad \blacksquare
\end{aligned} \tag{4.32}$$

Now for the functions $h$ constant on lattice cubes such that $h(x)=0$ for $x \notin \Lambda$ which were defined in Section 2, let $h_\alpha$ denote the value of $h(x)$ for $x$ in the cube $Q_\alpha$. Then we define $\delta(h)$ by

$$\delta(h) = \sum_{\alpha,\alpha'} (h_\alpha - h_{\alpha'})^2 \tag{4.33}$$

where the sum on $\alpha$, $\alpha'$ is over nearest neighbor cubes $Q_\alpha$, $Q_{\alpha'}$.

**Lemma 4.4.** Let $h$ be a discrete-valued function of Section 2 and $U \subset \Lambda$ a set such that every lattice cube $Q$ which intersects $\sum(h)$ is contained in $U$. Let $H_D$ be an operator as in Lemma 4.3 with potential $V(x)$ satisfying $0 \leqslant V(x) \leqslant 1/l_c^4$, $V(x)$ constant on $U$, $V(x) = V_0$, and suppose $L < 1$. Then there is a universal constant $C$ such that

$$\langle h, VH_D^{-1}(-\Delta_D+1)^2 H_D^{-1}Vh\rangle \geqslant CV_0^2(Ll_c)^7\,\delta(h) \tag{4.34}$$

*Proof.* We use the inequality (4.30) and choose the functions $\chi_i$ appropriately to get the right-hand side of (4.34). Suppose nearest neighbor cubes $Q_\alpha$ and $Q_{\alpha'}$ abut on a face which does not lie on $\partial\Lambda$. Then we choose $\chi_i$ to be a $C^\infty$ function supported in $Q_\alpha \cup Q_{\alpha'}$. Since

$$\int -\Delta_D\chi_i\, dx = 0 \tag{4.35}$$

it is clear we can choose $\chi_i$ such that

$$|\langle h, -\Delta_D\chi_i\rangle| \geqslant C_1(Ll_c)\,|h_\alpha - h_{\alpha'}| \tag{4.36}$$

$$\langle \chi_i, H_D^2\chi_i\rangle \leqslant C_2(Ll_c)^{-5} \tag{4.37}$$

where $C_1$ and $C_2$ are universal constants.

If $Q_\alpha$ and $Q_{\alpha'}$ abut on a face which does lie in $\partial\Lambda$ with $Q_\alpha$ contained in $\Lambda$, then we choose $\chi_i$ to be supported in $Q_\alpha$. Let $x^1$, $x^2$ be Cartesian coordinates in the face and $x^3$ be the coordinate at right angles with $x^3>0$ lying in $Q_\alpha$. We then put

$$\chi_i(x)=\chi(x^1, x^2)\,\zeta(x^3)\sin(x^3/Ll_c) \tag{4.38}$$

where $\chi(x^1, x^2)$ is a $C^\infty$ function with compact support in the face on $\partial\Lambda$, $\zeta(x^3)$ is a $C^\infty$ function such that $\zeta(x^3)=0$ if $x^3>Ll_c/2$, and $\zeta(x^3)=1$ if $x^3<Ll_c/4$. On choosing the functions $\chi$ and $\zeta$ appropriately in (4.38) we get inequalities (4.36) and (4.37) when $h$ has a discontinuity in $\partial\Lambda$. Summing over all possible $\chi_i$ then yields the inequality (4.34). ▮

**Lemma 4.5.** Let $h$ and $U$ be as in Lemma 4.4 and $L=(\beta l_c)^{1/10}$. Then for $\beta l_c<1$ there is the inequality

$$\begin{aligned}&\exp[-F(h)]E_{\mathfrak{L}_D}\{\exp[G_U(\psi+g(h))]\}\\&\quad\leqslant c_1\exp[c_2\,|U|-c_3(\beta l_c)^{-3/10}\,\delta(\beta^{1/2}h)]\end{aligned} \tag{4.39}$$

for universal constants $c_1$, $c_2$, $c_3$.

*Proof.* From Lemma 4.1 and (4.10) we have the inequality

$$\begin{aligned}\exp[G_U(\psi+g(h))]&=\exp[G_U(\psi+g(h)-h)]\\&\leqslant c_1\exp[\tfrac{1}{2}\langle\psi+g(h)-h,\ W(\psi+g(h)-h)\rangle]\end{aligned} \tag{4.40}$$

where $W$ is the operator

$$W=\frac{\gamma}{l_c^4}P\chi_U P+N \tag{4.41}$$

$\chi_U$ is the characteristic function of the set $U$, and $N$ is the operator $N=(2/l_c^4)(I-P)\chi_U(I-P)$. If we make the reverse transformation $\phi=\psi+g(h)$, then it is clear that the left-hand side of (4.39) is bounded by

$$\begin{aligned}&c_1\kappa(\mathfrak{L}_D)^{-1}E\left\{\exp\left[\frac{-1}{2l_c^4}\langle\phi-h,\phi-h\rangle+\frac{1}{2}\langle\phi-h,\ W(\phi-h)\rangle\right]\right\}\\&\quad\leqslant c_1\kappa(\mathfrak{L}_D)^{-1}E\left\{\exp\left[-\frac{1}{2}\langle\phi-h,\ V(\phi-h)\rangle\right.\right.\\&\qquad\left.\left.+\frac{1}{2}\langle\phi-h,\ N(\phi-h)\rangle\right]\right\}\end{aligned} \tag{4.42}$$

where $V$ is the potential

$$V(x)=\frac{1}{l_c^4}[1-\gamma\chi_U(x)] \tag{4.43}$$

We make the change of variable $\phi=\psi+g_V(h)$ in (4.42), where $g_V(h)$ is given by

$$g_V(h)=H_D^{-1}Vh \tag{4.44}$$

with $V$ the potential (4.43). If we define $F_V(h)$ by

$$\begin{aligned}2F_V(h)=\langle g_V(h), (-\Delta_D+1)^2\, g_V(h)\rangle\\ +\langle g_V(h)-h, V[g_V(h)-h]\rangle\end{aligned} \tag{4.45}$$

and $\kappa(H_D)$ by

$$\kappa(H_D)=\det[I+V(-\Delta_D+1)^{-2}]^{-1/2} \tag{4.46}$$

then the right-hand side of (4.42) is given by

$$\begin{aligned}c_1\kappa(H_D)\,\kappa(\mathfrak{L}_D)^{-1}\exp[-F_V(h)]\\ \times E_{H_D}\{\exp[\tfrac{1}{2}\langle\psi+g_V(h)-h, N(\psi+g_V(h)-h)\rangle]\}\end{aligned} \tag{4.47}$$

where $E_{H_D}$ denotes expectation with respect to the Gaussian field with covariance $H_D^{-1}$.

First we bound the expectation $E_{H_D}$ by

$$\begin{aligned}E_{H_D}\{\exp[\tfrac{1}{2}\langle\psi+g_V(h)-h, N(\psi+g_V(h)-h)\rangle]\}\\ \leqslant\exp[\langle g_V(h), Ng_V(h)\rangle]\cdot E_{H_D}[\exp(\langle\psi, N\psi\rangle)]\end{aligned} \tag{4.48}$$

Arguing as in Lemma 4.2, it is easy to see that if $L<\pi/2$, then

$$E_{H_D}[\exp(\langle\psi, N\psi\rangle)]\leqslant\exp(A_\gamma L^3|U|) \tag{4.49}$$

where $|U|$ is the number of lattice cubes contained in $U$, and $A_\gamma$ is a constant depending only on $\gamma<1$. It follows also in a similar fashion that if $L<\pi/2$, then

$$\tfrac{1}{2}F_V(h)-\langle g_V(h), Ng_V(h)\rangle\geqslant 0 \tag{4.50}$$

To estimate $\kappa(H_D)\,\kappa(\mathfrak{L}_D)^{-1}$, we observe that

$$\kappa(H_D)\,\kappa(\mathfrak{L}_D)^{-1}=\det\left(I-\frac{\gamma}{l_c^4}\chi_U\mathfrak{L}_D^{-1}\right)^{-1/2} \tag{4.51}$$

Now we may argue as in Lemma 4.2 to obtain the bound

$$\kappa(H_D)\,\kappa(\mathfrak{L}_D)^{-1} \leqslant \exp\left[\frac{\gamma A L^3}{2(1-\gamma)}\,|U|\right] \tag{4.52}$$

We conclude then from (4.48)–(4.52) that the right-hand side of (4.42) is bounded by

$$c_1 \exp[-\tfrac{1}{2}F_V(h)]\exp(c_2\,|U|) \tag{4.53}$$

for constants $c_1$ and $c_2$.

Finally, we apply Lemma 4.4 to obtain the inequality

$$\begin{aligned}\frac{1}{2}F_V(h) &\geqslant \frac{1}{2}\,C\,\frac{(1-\gamma)^2}{l_c^8}\,(Ll_c)^7\,\delta(h)\\ &= \frac{1}{2}\,C(1-\gamma)^2\,(\beta l_c)^{-3/10}\,\delta(\beta^{1/2}h)\end{aligned} \tag{4.54}$$

This last inequality taken together with (4.53) completes the proof. ∎

**Lemma 4.6.** Let $N$ be an integer, $N \geqslant 2$, and $L = (\beta l_c)^{1/10}$. Then there exist constants $C > 0$, $\delta > 0$ such that if $\beta l_c < \delta$, then

$$\sum_{\substack{Q \in X\\ |X| = N}} |K_1(X)| \leqslant \exp[-CN(\beta l_c)^{-3/10}] \tag{4.55}$$

*Proof.* From (3.5) and Lemma 4.5 we have the inequality

$$|K_1(X)| \leqslant c_1 \exp(c_2\,|X|) \sum_{h_X} \exp[-c_3(\beta l_c)^{-3/10}\,\delta(\beta^{1/2}h_X)] \tag{4.56}$$

It is easy to see that for $\beta l_c$ small there is a constant $c_4 > 0$ such that

$$\sum_{h_X} \exp[-c_3(\beta l_c)^{-3/10}\,\delta(\beta^{1/2}h_X)] \leqslant \exp[-c_4\,|X|\,(\beta l_c)^{-3/10}] \tag{4.57}$$

The result follows now from the well-known fact that the number of connected sets $X$ with $|X| = N$ containing a given cube $Q$ is bounded by $80^N$. ∎

## 5. ESTIMATION OF TWO-PARTICLE ACTIVITIES

We turn next to estimating two-particle activities $K_2(X)$ for different sets $X$. From here on we shall always take $L = (\beta l_c)^{1/10}$. First we consider the case when $|X| = 2$.

**Lemma 5.1**. There is a constant $C$ such that if $\beta l_c < 1$, then for any lattice cubes $Q$, $Q'$

$$|K_2(Q, Q')| \leqslant CL^3(\beta l_c)^{1/5} \exp[-d(Q, Q')/\lambda l_c] \tag{5.1}$$

*Proof.* Now $K_2(Q, Q')$ is given by (3.8). We use the formulas (4.7) and (4.9) to obtain the identity

$$\begin{aligned}
&\int_Q f(y) \frac{\delta}{\delta\psi(y)} dy \exp[G_Q(\psi)] \\
&\quad = \frac{d}{dt} \exp[G_Q(\psi + tf)]|_{t=0} \\
&\quad = \exp[R_Q(\psi)] \left\{ \frac{L^{3/2}}{l_c^{1/2}} (Pf)\, r'_\eta \left( \frac{L^{3/2} P\psi}{l_c^{1/2}} \right) \right. \\
&\qquad + \frac{1}{l_c^4} \int_Q [\psi(x) - P\psi(x)][f(x) - Pf(x)] \\
&\qquad \times \left( 1 - 2 \int_0^1 \int_0^1 u \cos\{\beta^{1/2} P\psi(x) + uv\beta^{1/2}[\psi(x) - P\psi(x)]\}\, du\, dv \right) dx \\
&\qquad + \frac{\sqrt{\beta}}{l_c^4} \int_Q [\psi(x) - P\psi(x)]^2 \int_0^1 \int_0^1 u[Pf(x) + uv(I - P) f(x)] \\
&\qquad \left. \times \sin\{\beta^{1/2} P\psi(x) + uv\beta^{1/2}[\psi(x) - P\psi(x)]\}\, du\, dv\, dx \right\} \\
&\quad = \exp[R_Q(\psi)]\{I_1(\psi) + I_2(\psi) + I_3(\psi)\}
\end{aligned} \tag{5.2}$$

From Lemma 4.1 we have

$$|I_1(\psi)| \leqslant c_1 L^{3/2} \eta^{1/3} \sup_{y \in Q} |f(y)/l_c^{1/2}| \exp(\gamma A^2/2) \tag{5.3}$$

where $c_1$ is a constant and $\eta$ and $A$ are given by (4.8). On using the Schwarz inequality, it follows that

$$\begin{aligned}
|I_2(\psi)| &\leqslant \frac{2}{l_c^4} \sqrt{3}\, L l_c \sup_{y \in Q} |f'(y)| \int_Q |\psi(x) - P\psi(x)|\, dx \\
&\leqslant \frac{2}{l_c^2} \sqrt{3}\, L l_c \sup_{y \in Q} |f'(y)|\, (\text{Vol } Q)^{1/2} \\
&\quad \times \left\{ \frac{1}{l_c^4} \int_Q [\psi(x) - P\psi(x)]^2\, dx \right\}^{1/2} \\
&\leqslant 2\sqrt{3}\, L^{3/2} (\beta l_c)^{1/10} \sup_{y \in Q} |l_c^{1/2} f'(y)| \\
&\quad \times \exp\left\{ \frac{1}{l_c^4} \int_Q [\psi(x) - P\psi(x)]^2\, dx \right\}
\end{aligned} \tag{5.4}$$

It is immediately evident that

$$|I_3(\psi)| \leqslant L^{3/2}(\beta l_c)^{1/10} \sup_{y \in Q} |f(y)/l_c^{1/2}|$$
$$\times \exp\left\{\frac{1}{l_c^4}\int_Q [\psi(x) - P\psi(x)]^2\, dx\right\} \tag{5.5}$$

Now let $W_X$ be the operator

$$W_X = \frac{\gamma}{l_c^4} P\chi_X P + \frac{4}{l_c^4}(I-P)\,\chi_X(I-P) \tag{5.6}$$

From (3.8), Lemma 2.1, and (5.3)–(5.5) it follows now that there is a constant $C$ such that

$$|K_2(Q, Q')| \leqslant CL^3(\beta l_c)^{1/5} \exp[-d(Q, Q')/\lambda l_c]$$
$$\times \int_0^1 ds E_{\mathfrak{L}_{D_s}}\left[\exp\left(\frac{1}{2}\langle\psi, W_X\psi\rangle\right)\right] \tag{5.7}$$

where $X = Q \cup Q'$.

We estimate the expectation in (5.7) by using the technique of ref. 2. Thus, we write

$$\exp\left(\frac{1}{2}\langle\psi, W_X\psi\rangle\right) = \int \exp(-\langle f, \psi\rangle)\, d\mu(f) \tag{5.8}$$

where $d\mu(f)$ is a Gaussian probability distribution. Then we have

$$E_{\mathfrak{L}_{D_s}}[\exp(\tfrac{1}{2}\langle\psi, W_X\psi\rangle)]$$
$$= \int d\mu(f)\, E_{\mathfrak{L}_{D_s}}[\exp(-\langle f, \psi\rangle)]$$
$$= \int d\mu(f) \exp\left(\frac{1}{2}\langle f, C_{D,s} f\rangle\right)$$
$$\leqslant \left[\int d\mu(f) \exp\left(\frac{1}{2}\langle f, C_{D,1} f\rangle\right)\right]^s$$
$$\left[\int d\mu(f) \exp\left(\frac{1}{2}\langle f, C_{D,0} f\rangle\right)\right]^{1-s} \tag{5.9}$$

by Holder's inequality. Now we reverse the process and write

$$\int d\mu(f) \exp\left(\frac{1}{2} \langle f, C_{D,1} f \rangle\right) = E_{\mathfrak{Q}_D}\left[\exp\left(\frac{1}{2} \langle \psi, W_X \psi \rangle\right)\right] \tag{5.10}$$

$$\int d\mu(f) \exp\left(\frac{1}{2} \langle f, C_{D,0} f \rangle\right) = E_{\mathfrak{Q}_D}\left[\exp\left(\frac{1}{2} \langle \psi, W_Q \psi \rangle\right)\right]$$
$$\times E_{\mathfrak{Q}_D}\left[\exp\left(\frac{1}{2} \langle \psi, W_{Q'} \psi \rangle\right)\right] \tag{5.11}$$

We have already shown how to estimate the quantities on the right in (5.10) and (5.11). They are bounded by a constant. The result now follows. ▌

**Lemma 5.2.** Let $Y$ be a connected set with $|Y| \geqslant 2$, and $Q$ denote a lattice cube, $Q \notin Y$. Then there exist constants $c_1$, $c_2$, $\delta > 0$ such that if $\beta l_c < \delta$, then

$$|K_2(Y, Q)| \leqslant c_1 L^3 \exp[-d(Y, Q)/\lambda l_c](\beta l_c)^{-1/2}$$
$$\times \exp[-c_2 |Y| (\beta l_c)^{-3/10}] \tag{5.12}$$

*Proof.* We use the formula (3.10) for $K_2(Y, Q)$. Letting $X = Y \cup Q$, then

$$K_2(Y, Q) = \sum_{h_Y} \exp[-F(h_Y)] \int_0^1 ds[I(s, h_Y) + J(s, h_Y)] \tag{5.13}$$

where

$$I(s, h) = E_{\mathfrak{Q}_{D_s}}\left\{\int_Y \int_Q C_D(x, y)\right.$$
$$\left.\times \frac{\delta}{\delta\psi(x)} \frac{\delta}{\delta\psi(y)} dx\, dy \exp[G_X(\psi + g_s)]\right\} \tag{5.14}$$
$$J(s, h) = E_{\mathfrak{Q}_{D_s}}\left\{\int_Q g(h)(x) \frac{\delta}{\delta\psi(x)} dx \exp[G_X(\psi + g_s)]\right\}$$

From Lemma 5.1 it is easy to see that

$$|I(s, h)| \leqslant CL^3(\beta l_c)^{1/5} |Y| \exp[-d(Y, Q)/\lambda l_c]$$
$$\times E_{\mathfrak{Q}_{D_s}}\{\exp[\tfrac{1}{2}\langle \psi + g_s - h, W_X(\psi + g_s - h)\rangle]\} \tag{5.15}$$

where $C$ is the same constant as in (5.7). Using the Gaussian representation (5.8) from Lemma 5.1, we have the inequality

$$\begin{aligned} &E_{\mathfrak{Q}_{D_s}}\{\exp[\tfrac{1}{2}\langle\psi+g_s-h,\, W_X(\psi+g_s-h)\rangle]\} \\ &\quad\leqslant E_{\mathfrak{Q}_D}\{\exp[\tfrac{1}{2}\langle\psi+g-h,\, W_X(\psi+g-h)\rangle]\}^s \\ &\quad\quad\times(E_{\mathfrak{Q}_D}\{\exp[\tfrac{1}{2}\langle\psi+g-h,\, W_Y(\psi+g-h)\rangle]\} \\ &\quad\quad\times E_{\mathfrak{Q}_D}[\exp(\tfrac{1}{2}\langle\psi,\, W_Q\psi\rangle)])^{1-s} \end{aligned} \tag{5.16}$$

Arguing as in Lemmas 4.2 and 4.5, we conclude from (5.16) that

$$\begin{aligned} &\exp[-F(h)]\, E_{\mathfrak{Q}_{D_s}}\{\exp[\tfrac{1}{2}\langle\psi+g_s-h,\, W_X(\psi+g_s-h)\rangle]\} \\ &\quad\leqslant c_1\exp[c_2\,|X|-c_3(\beta l_c)^{-3/10}\,\delta(\beta^{1/2}h)] \end{aligned} \tag{5.7}$$

We bound $J(s,h)$ in a similar way using Lemma 5.1 to obtain the inequality

$$\begin{aligned} &|J(s,h)|\leqslant CL^3(\beta l_c)^{-1/2}\,|Y|\exp[-d(Y,Q)/\lambda l_c] \\ &\quad\times\delta(\beta^{1/2}h)^{1/2}\, E_{\mathfrak{Q}_{D_s}}\{\exp[\tfrac{1}{2}\langle\psi+g_s-h,\, W_X(\psi+g_s-h)\rangle]\} \end{aligned} \tag{5.18}$$

To see this, we take $f(x)=g(h)(x)$ in (5.2). Thus, we need to establish the estimate

$$\begin{aligned} &\sup_{x\in Q}\,[|g(h)(x)/l_c^{1/2}|+|l_c^{1/2}\nabla g(h)(x)|] \\ &\quad\leqslant CL^{3/2}(\beta l_c)^{-1/2}\,|Y|\exp[-d(Y,Q)/\lambda l_c]\,\delta(\beta^{1/2}h)^{1/2} \end{aligned} \tag{5.19}$$

By Lemma 2.1 we have for $x\in Q$,

$$\begin{aligned} \left|\frac{g(h)(x)}{l_c^{1/2}}\right| &\leqslant\frac{A}{l_c^{7/2}}\int_Y\exp\left(\frac{-|x-y|}{\lambda l_c}\right)|h(y)|\,dy \\ &\leqslant\frac{A}{l_c^{7/2}}\left[\int_Y\exp\left(-\frac{6\,|x-y|}{5\lambda l_c}\right)dy\right]^{5/6}\|h\|_6 \\ &\leqslant\frac{AL^3}{l_c^{1/2}}\,|Y|\exp\left(\frac{-d(Y,Q)}{\lambda l_c}\right)\left[\sum_\alpha h_\alpha^6\right]^{1/6} \end{aligned} \tag{5.20}$$

Now we use the discrete Sobolev inequality,

$$\left[\sum_\alpha h_\alpha^6\right]^{1/6}\leqslant K\,\delta(h)^{1/2} \tag{5.21}$$

for some universal constant $K$, which, together with (5.20), yields the inequality (5.19) for the first term on the left in (5.19). The inequality for the second term follows in a similar fashion using Lemma 2.1 again. The result follows easily now from (5.15)–(5.19). ▌

**Lemma 5.3.** Let $Y$ and $Z$ be disjoint connected sets with $|Y| \geqslant 2$, $|Z| \geqslant 2$. Then there exist constants $c_1$, $c_2$, $\delta > 0$ such that if $\beta l_c < \delta$, then

$$|K_2(Y, Z)| \leqslant c_1 L^3 \exp[-d(Y, Z)/2\lambda l_c](\beta l_c)^{-1} \times \exp[-c_2 |Y \cup Z| (\beta l_c)^{-3/10}] \tag{5.22}$$

*Proof.* We use the formula (3.13) for $K_2(Y, Z)$. Thus we write

$$K_2(Y, Z) = \sum_{h_Y, h_Z} \exp(-F_s) \int_0^1 ds[I(s, h_Y, h_Z) + J(s, h_Y, h_Z) + M(s, h_Y, h_Z)] \tag{5.23}$$

where

$$\begin{aligned} I(s, h_Y, h_Z) &= E_{\mathfrak{Q}_{D_s}} \left\{ \int_Y \int_Z C_D(x, y) \times \frac{\delta}{\delta\psi(x)} \frac{\delta}{\delta\psi(y)} dx\, dy \exp[G_X(\psi + g_s)] \right\} \\ J(s, h_Y, h_Z) &= E_{\mathfrak{Q}_{D_s}} \left\{ \left[ \int_Y g(h_Z)(x) \frac{\delta}{\delta\psi(x)} dx + \int_Z g(h_Y) \frac{\delta}{\delta\psi(x)} dx \right] \exp[G_X(\psi + g_s)] \right\} \\ M(s, h_Y, h_Z) &= -2F(h_Y, h_Z)\, E_{\mathfrak{Q}_{D_s}}\{\exp[G_X(\psi + g_s)]\} \end{aligned} \tag{5.24}$$

Here $X$ is the set $X = Y \cup Z$ and $g_s$ interpolates $h = h_Y + h_Z$. We can bound $I$ and $J$ in a similar way to Lemma 5.2. In fact, we can easily see that

$$|I(s, h_Y, h_Z)| \leqslant CL^3(\beta l_c)^{1/5} |Y| \cdot |Z| \exp[-d(Y, Z)/\lambda l_c] \times E_{\mathfrak{Q}_{D_s}}\{\exp[\tfrac{1}{2}\langle \psi + g_s - h, W_X(\psi + g_s - h)\rangle]\} \tag{5.25}$$

where $C$ is the same constant as in (5.7), and $J$ satisfies the inequality

$$|J(s, h_Y, h_Z)| \leqslant CL^3 |Y| \cdot |Z| (\beta l_c)^{-1/2} \times \exp[-d(Y, Z)/\lambda l_c]\, \delta(\beta^{1/2} h)^{1/2} \times E_{\mathfrak{Q}_{D_s}}\{\exp[\tfrac{1}{2}\langle \psi + g_s - h, W_X(\psi + g_s - h)\rangle]\} \tag{5.26}$$

To bound $M$, we need to estimate $F(h_Y, h_Z)$, which is defined in (2.13). Concentrating on the first term in (2.13) and using Lemma 2.1, we have the inequality

$$
\begin{aligned}
&|\langle g(h_Y), (-\Delta_D+1)^2\, g(h_Z)\rangle| \\
&\quad = |\langle (-\Delta_D+1)\, g(h_Y), (-\Delta_D+1)\, g(h_Z)\rangle| \\
&\quad \leqslant \frac{A^2}{l_c^{10}} \int_Y \int_\Lambda \int_Z |h_Y(y)| \exp\left(-\frac{|y-x|}{\lambda l_c} - \frac{|x-z|}{\lambda l_c}\right) |h_Z(z)|\, dy\, dx\, dz \\
&\quad \leqslant \frac{C_1}{l_c^7} \int_Y \int_Z |h_Y(y)| \exp\left(-\frac{|y-z|}{2\lambda l_c}\right) |h_Z(z)|\, dy\, dz \\
&\quad \leqslant \frac{C_1}{l_c^7} [\|h_Y\|_6 (L^3 l_c^3 |Z|)^{1/6}][\|h_Z\|_6 (L^3 l_c^3 |Y|)^{1/6}] \\
&\qquad \times \left[\int_Y \int_Z \exp\left(\frac{-3\,|y-z|}{4\lambda l_c}\right) dy\, dz\right]^{2/3} \qquad (5.27)
\end{aligned}
$$

by the Hölder inequality. If we now apply the discrete Sobolev inequality, we have then

$$
\begin{aligned}
&|\langle g(h_Y), (-\Delta_D+1)^2\, g(h_Z)\rangle| \\
&\quad \leqslant C_2\, |Y| \cdot |Z|\, L^6 (\beta l_c)^{-1}\, \delta(\beta^{1/2} h_Y)^{1/2} \\
&\qquad \times \delta(\beta^{1/2} h_Z)^{1/2} \exp[-d(Y,Z)/2\lambda l_c] \qquad (5.28)
\end{aligned}
$$

In a similar way we have

$$
\begin{aligned}
&\frac{1}{l_c^4} |\langle g(h_Y), h_Z\rangle| \leqslant C_3\, |Y| \cdot |Z|\, L^6 (\beta l_c)^{-1}\, \delta(\beta^{1/2} h_Y)^{1/2} \\
&\qquad \times \delta(\beta^{1/2} h_Z)^{1/2} \exp[-d(Y,Z)/\lambda l_c] \qquad (5.29)
\end{aligned}
$$

The inequalities (5.28) and (5.29) will yield a bound on $F(h_Y, h_Z)$ and hence we conclude that

$$
\begin{aligned}
&|M(s, h_Y, h_Z)| \leqslant C L^3\, |Y| \cdot |Z|\, (\beta l_c)^{-1} \exp[-d(Y,Z)/2\lambda l_c]\, \delta(\beta^{1/2} h) \\
&\qquad \times E_{\mathfrak{Q}_{D_s}}\{\exp[\tfrac{1}{2}\langle \psi + g_s - h, W_X(\psi + g_s - h)\rangle]\} \qquad (5.30)
\end{aligned}
$$

We estimate $E_{\mathfrak{Q}_{D_s}}$ as in (5.16) to obtain the inequality

$$
\begin{aligned}
&\exp(-F_s)\, E_{\mathfrak{Q}_{D_s}}\{\exp[\tfrac{1}{2}\langle \psi + g_s - h, W_X(\psi + g_s - h)\rangle]\} \\
&\quad \leqslant (\exp[-F(h)] \\
&\qquad \times E_{\mathfrak{Q}_D}\{\exp[\tfrac{1}{2}\langle \psi + g(h) - h, W_X(\psi + g(h) - h)\rangle]\})^s \\
&\qquad \times (\exp[-F(h_Y)] \\
&\qquad \times E_{\mathfrak{Q}_D}\{\exp[\tfrac{1}{2}\langle \psi + g(h_Y) - h_Y, W_Y(\psi + g(h_Y) - h_Y)\rangle]\})^{1-s} \\
&\qquad \times (\exp[-F(h_Z)] \\
&\qquad \times E_{\mathfrak{Q}_D}\{\exp[\tfrac{1}{2}\langle \psi + g(h_Z) - h_Z, W_Z(\psi + g(h_Z) - h_Z)\rangle]\})^{1-s} \qquad (5.31)
\end{aligned}
$$

The three terms in large parentheses on the right in (5.31) may be estimated from Lemma 4.5 and hence we conclude that (5.31) is bounded above by

$$c_1 \exp[c_2 |X| - c_3(\beta l_c)^{-3/10} \delta(\beta^{1/2}h)] \tag{5.32}$$

The inequality (5.22) easily follows now by summing with respect to $h_Y$ and $h_Z$ using the estimates (5.25), (5.26), (5.30), and (5.32). ∎

**Lemma 5.4.** Let $N$ be an integer, $N \geqslant 3$. Then there exist constants $C > 0$ and $\delta > 0$ such that if $\beta l_c < \delta$, then

$$\sum_{\substack{Q \in X \\ |X| = 2}} |K_2(X)| \leqslant C^{-1}(\beta l_c)^{1/5} \tag{5.33}$$

$$\sum_{\substack{Q \in X \\ |X| = N}} |K_2(X)| \leqslant \exp[-CN(\beta l_c)^{-3/10}] \tag{5.34}$$

*Proof.* The inequality (5.33) follows from (5.1) on summing with respect to $Q'$. The inequality (5.34) follows from (5.12) and (5.22). We need to show that

$$\sum_{\substack{Q \in Y \cup Q' \\ |Y| = N-1}} |K_2(Y, Q')| \leqslant \exp[-CN(\beta l_c)^{-3/10}] \tag{5.35}$$

To see this, we first consider

$$\begin{aligned}
\sum_{|Y| = N-1} |K_2(Y, Q)| &\leqslant c_1 L^3 \sum_{Y} \sum_{Q' \in Y} \exp[-d(Q', Q)/\lambda l_c] \\
&\qquad \times (\beta l_c)^{-1/2} \exp[-c_2 |Y| \cdot (\beta l_c)^{-3/10}] \\
&= c_1 L^3 \sum_{Q'} \sum_{Q' \in Y} \exp[-d(Q', Q)/\lambda l_c] \\
&\qquad \times (\beta l_c)^{-1/2} \exp[-c_2 |Y| \, (\beta l_c)^{-3/10}] \\
&\leqslant c_1 \exp[-c_3 N(\beta l_c)^{-3/10}] \\
&\qquad \times \sum_{Q'} L^3 \exp[-d(Q', Q)/\lambda l_c] \\
&\leqslant c_4 \exp[-c_3 N(\beta l_c)^{-3/10}]
\end{aligned} \tag{5.36}$$

Next we need to consider

$$\sum_{Q \in Y, |Y| = N-1, Q'} K_2(Y, Q')$$

$$\leqslant c_1 L^3 \sum_{Q \in Y} \sum_{Q'' \in Y, Q'} \exp[-d(Q'', Q')/\lambda l_c]$$

$$\times (\beta l_c)^{-1/2} \exp[-c_2 |Y| (\beta l_c)^{-3/10}]$$

$$\leqslant c_3 \sum_{Q \in Y} |Y| (\beta l_c)^{-1/2} \exp[-c_2 |Y| (\beta l_c)^{-3/10}]$$

$$\leqslant c_4 \exp[-c_5 N(\beta l_c)^{-3/10}] \tag{5.37}$$

The inequalities (5.36) and (5.37) prove (5.35). In a similar way we can deal with the case of $K_2(Y, Z)$ using (5.22). ∎

## 6. PROOF OF THEOREM 1.2

In this section we shall prove Theorem 1.2 by showing that the cluster expansion defined in Section 3 is convergent and is term-by-term differentiable in $\beta$, $z$. Our first lemma bounds the $n$-particle activities defined by (3.19). The proof is similar to the proofs in Section 5 bounding the two-particle activities. We shall merely state the result.

**Lemma 6.1.** Let $K_{n,\Lambda}(X_1, X_2, ..., X_n)$ be defined by (3.19) and put $X = \bigcup_{i=1}^{n} X_i$. Let $D^r$ denote differentiation of order $r$ in the variables $\beta$, $z$. Then there exist universal constants $c_2$, $m$, $\varepsilon > 0$ and a constant $c_1(\beta, z, r)$ depending on $\beta$, $z$, $r$ such that

$$|D^r K_{n,\Lambda}(X_1, X_2, ..., X_n)|$$

$$\leqslant c_1(\beta, z, r) L^{3(n-1)} (\beta l_c)^{n/10} \exp[-c_2(|X| - n)(\beta l_c)^{-3/10}]$$

$$\times \sum_T \prod_{i \in T} [n_T(i)!]^m \sum_{(i,j) \in T} \exp[-d(X_i, X_j)/2\lambda l_c] \tag{6.1}$$

Here $T$ is a tree graph on $1, 2, ..., n$ and $n_T(i)$ is the number of bounds which intersect the vertex $i$, $1 \leqslant i \leqslant n$.

Also in (6.1) the differentiation is taken under the assumption that $L$ is fixed.

Our second lemma concerns the existence of the thermodynamic limit.

**Lemma 6.2.** With $K_{n,\Lambda}$ as in Lemma 6.1, the limit

$$\lim_{\Lambda \to \infty} K_{n,\Lambda}(X_1, X_2,..., X_n) = K_n(X_1,..., X_n) \tag{6.2}$$

exists and is translation invariant. Further, for any $D^r$,

$$\lim_{\Lambda \to \infty} D^r K_{n,\Lambda}(X_1, X_2, ..., X_n) = D^r K_n(X_1, ..., X_n) \tag{6.3}$$

*Proof.* This can be accomplished as in ref. 2 by introducing a covariance in the definition of $K_{n,\Lambda}$ which interpolates $\mathfrak{L}_D$ and the free operator $\mathfrak{L}_0$ corresponding to $\Lambda = \mathbf{R}^3$. The interpolated covariance is

$$\mathfrak{L}_t = t\mathfrak{L}_D + (1-t)\mathfrak{L}_0 \tag{6.4}$$

and with corresponding $K_n$ denoted $K_{n,t}$. Thus,

$$K_{n,1} = K_{n,\Lambda}, \qquad K_{n,0} = K_n \tag{6.5}$$

Now the difference $K_{n,1} - K_{n,0}$ can be computed using the fundamental theorem of calculus and it is a standard procedure to estimate it.

*Proof of Theorem* 1.2. This follows in a standard fashion on expanding out (3.21) as in ref. 1 and using Lemmas 6.1 and 6.2.

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